On the Conformal Geometry of Transverse Riemann-Lorentz Manifolds

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Abstract

Physical reasons suggested in [2] for the *Quantum Gravity Problem* lead us to study *type-changing metrics* on a manifold. The most interesting cases are *Transverse Riemann-Lorentz Manifolds*. Here we study the conformal geometry of such manifolds.

1 Preliminaries

Let M be a connected manifold, $\dim M = m \geq 2$, and let g be a symmetric covariant tensor field of order 2 on M. Assume that the set Σ of points where g degenerates is not empty. Consider $p \in \Sigma$ and $(\mathbb{U}, x)$ a coordinate system around p. We say that g is a *transverse type-changing metric* on p if $d_p (\det (g_{ab})) \neq 0$ (this condition does not depend on the choice of the coordinates). We call (M, g) *transverse type-changing pseudoriemannian manifold* if g is transverse type-changing on every point of Σ. In this case, Σ is a hypersurface of M. Moreover, at every point of Σ there exists a one-dimensional *radical*, that is the subspace $Rad_p (M)$ of T_pM which is *g-ortogonal to the whole* T_pM (and it can be transverse or tangent to the hypersurface Σ). The *index* of g is constant on every connected component of $\mathbb{M} = M - \Sigma$, thus $\mathbb{M}$ is a union of connected pseudoriemannian manifolds. Locally, Σ separates two pseudoriemannian manifolds whose indices differ in one unit (so we call

Σ *transverse type-changing hypersurface*, in particular Σ is orientable). The most interesting cases are those in which Σ separates a riemannian part from a lorentzian one. We call these cases *transverse Riemann-Lorentz manifolds.*

Let $\tau \in C^{\infty}(M)$ be such that $\tau \mid_{\Sigma} = 0$ and $d\tau \mid_{\Sigma} \neq 0$. We say that (locally, around Σ) $\tau = 0$ *is an equation for* Σ. Given $f \in C^{\infty}(M)$, it holds: $\tau \mid_{\Sigma} = 0 \Leftrightarrow f = k\tau$, for some $k \in C^{\infty}(M)$. In what follows we shall use this fact extensively.

On $\mathbb{M}$ we have naturally defined all the objects associated to pseudoriemannian geometry, derived from the Levi-Civita connection. In [4], [5], [6], [7] and [1], the extendibility of geodesics, parallel transport and curvatures have been studied. Our aim in the present paper is to study the conformal geometry of transverse Riemann-Lorentz manifolds, including criteria for the extendibility of the *Weyl conformal curvature.*

Let (M, g) be a transverse Riemann-Lorentz manifold. First of all, note that we do not have any Levi-Civita connection ∇ defined on the whole M. However we have ([4]) a unique torsion-free metric *dual connection*

$$\square : \mathfrak{X}\left(M\right) \times \mathfrak{X}\left(M\right) \to \mathfrak{X}^{*}\left(M\right)$$

on M defined by a *Koszul-like formula.* On $\mathbb{M}$ it holds $\square_X Y\left(Z\right) = g\left(\nabla_X Y, Z\right)$, and thus the concepts derived from Levi-Civita connection ∇ (on $\mathbb{M}$) coincide with those derived from the dual connection $\square$.

We say that a vectorfield $R \in \mathfrak{X}\left(M\right)$ *is radical* if $R_p \in Rad_p\left(M\right) - \{0\}$ for all $p \in \Sigma$. Given a radical vectorfield $R \in \mathfrak{X}\left(M\right)$, $\square_X Y\left(R\right)|_{\Sigma}$ only depends on $X|_{\Sigma}$ and $Y|_{\Sigma}$, thus we obtain the following well-defined map

$$II^{R} : \mathfrak{X}_{\Sigma} \times \mathfrak{X}_{\Sigma} \to C^{\infty}\left(\Sigma\right), \left(X, Y\right) \mapsto \square_X Y\left(R\right)$$

Note that the II^{R}-orthogonal complement to $Rad_p\left(M\right)$ is $T_p\Sigma$ ([7], 1(a)), thus $X \in \mathfrak{X}_{\Sigma}$ is tangent to Σ if and only if $II^{R}\left(X, R\right) = 0$.

Because of the properties of $\square$, the restriction of II^{R} to vectorfields in $\mathfrak{X}\left(\Sigma\right)$ is a well-defined $(0,2)$ symmetric tensor field $II^{R}_{\Sigma} \in S^{2}\left(\Sigma\right)$. Furthermore, since $\square_X Y$ is a one-form on M and the radical is one-dimensional, the condition $II^{R}_{\Sigma} = 0$ does not depend on the radical vectorfield R. *A transverse Riemann-Lorentz manifold is said to be II-flat if* $II^{R}_{\Sigma} = 0$*, for some (and thus, for any) radical vectorfield* R. It turns out ([7] for transverse, [1] for tangent radical) that M is II-flat if and only if all covariant derivatives $\nabla_X Y$, for $X, Y \in \mathfrak{X}\left(M\right)$ tangent to Σ, smoothly extend to M. Moreover, in that case, $\nabla_X Y|_\Sigma$ only depends on $X|_\Sigma$ and $Y|_\Sigma$, thus we obtain another well-defined map

$$III^R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma), (X, Y) \mapsto II^R(\nabla_X Y, R)$$

which is a $(0,2)$ symmetric tensorfield on Σ. *A transverse Riemann-Lorentz II-flat metric is said to be III-flat if $III^R = 0$.*

If the radical is tangent, $\nabla_R R$ becomes transverse ([1]); therefore, in order that a II-flat metric becomes III-flat, the radical must be transverse. And we have the following result ([7]), concerning the extendibility of curvature tensors:

Theorem 1 *The covariant curvature K smoothly extends to M if and only if the radical is transverse and g is II-flat, while the Ricci tensor Ric smoothly extends to M if and only if the radical is transverse and g is III-flat.*

2 A Gauss formula for Transverse Riemann-Lorentz Manifolds

Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical.

Lemma 2 *There exists a unique (canonically defined) radical vectorfield R such that $II^R(R, R) = 1$.*

Proof: Given a radical vectorfield U, consider $R = \left(II^U(U, U)\right)^{-\frac{1}{3}} \cdot U$, which is a well-defined radical vectorfield (since the radical is transverse). Thus $II^R(R, R) = 1$. Furthermore, if $Z = fR$ is another radical vectorfield such that $II^Z(Z, Z) = 1$, then $1 = II^Z(Z, Z) = f^3 II^R(R, R) = f^3$, and consequently $f = 1$ ♣

Suppose that (M, g) is II-flat. As we said before, given $X, Y \in \mathfrak{X}(\Sigma)$, $\nabla_X Y$ is well-defined. Moreover, $\tan(\nabla_X Y) := \nabla_X Y - III^R(X, Y) \cdot R$ is indeed tangent to Σ, since

$$II^R(R, \tan(\nabla_X Y)) = III^R(X, Y) - III^R(X, Y)\, II^R(R, R) = 0$$

Lemma 3 *If $X, Y \in \mathfrak{X}(\Sigma)$ and ∇^Σ is the Levi-Civita connection of (Σ, g_Σ), it holds*

$$\nabla_X Y = \nabla^\Sigma_X Y + III^R(X, Y) \cdot R$$

Proof: Let be $Z \in \mathfrak{X}(\Sigma)$. Since (M, g) is II-flat, $\nabla_X Y$ is well defined and it must hold $\Box_X Y(Z) = g(\nabla_X Y, Z) = g_\Sigma(\tan(\nabla_X Y), Z)$. On the other hand, $\Box$ has always a good restriction $\Box : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^*(\Sigma)$, which must coincide with $\Box^\Sigma$, the unique torsion-free metric dual connection on (Σ, g_Σ). Since (Σ, g_Σ) is riemannian, it must hold $\Box^\Sigma_X Y(Z) = g_\Sigma\left(\nabla^\Sigma_X Y, Z\right)$, and the result follows ♣

The existence of a canonical radical vectorfield leads to the following Gauss formula.

Proposition 4 *Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical and II-flat. Then Σ is "totally geodesic" in the sense that, if $X, Y, Z, T \in \mathfrak{X}(\Sigma)$ it holds*

$$K(X, Y, Z, T) = K^\Sigma(X, Y, Z, T)$$

where K^Σ is the covariant curvature of Σ.

Proof: As we said in the proof of previous lemma we have, for $X, Y, Z, T \in \mathfrak{X}(\Sigma)$: $\Box_X Y(Z) = \Box^\Sigma_X Y(Z)$, where $\Box^\Sigma$ is the dual connection of (Σ, g_Σ). Moreover, since $\Box_X R(T) = -\Box_X T(R) = -II^R(X, T) = 0$, again previous lemma leads to

$$\Box_X(\nabla_Y Z)(T) = \Box_X\left(\nabla^\Sigma_Y Z + III^R(Y, Z) R\right)(T) = \Box^\Sigma_X\left(\nabla^\Sigma_Y Z\right)(T)$$

what gives the result ♣

Corollary 5 *Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical. If $(\mathbb{M}, g)$ is flat, then (M, g) is III-flat and Σ is flat.*

Proof: If $K = 0$ then $Ric = 0$. In particular, Ric extends to M, thus by Theorem 1, (M, g) is III-flat. By Proposition 4, Σ is flat ♣

We now restate Theorem 9 of [5] in the following terms (the flatness of Σ, being a consequence of the Collorary, needs not be included as an extra hypothesis):

Theorem 6 *Let (M, g) be a transverse Riemann-Lorentz manifold. Then, M is locally flat around Σ if and only if, around every singular point $p \in \Sigma$, there exists a coordinate system $(\mathbb{U}, x)$ such that $g = \sum_{i=0}^{m-1}\left(dx^i\right)^2 + \tau\left(dx^m\right)^2$, where $\tau = 0$ is a local equation for Σ.*

3 Conformal geometry and the extendibility of Weyl curvature

Let us consider a transverse Riemann-Lorentz manifold (M, g) and the family $\mathcal{C} = \left\{e^{2f}g : f \in C^{\infty}(M)\right\}$. Take $\overline{g} = e^{2f}g \in \mathcal{C}$. Then $(M, \overline{g})$ is also a transverse Riemann-Lorentz manifold, and $\overline{\Sigma} = \Sigma$. Moreover, for each singular point $p \in \Sigma$ the radical subspaces are the same: $\overline{Rad}_p(M) = Rad_p(M)$. We say that $(M, \mathcal{C})$ *is a transverse Riemann-Lorentz conformal manifold if some (and thus any)* $g \in \mathcal{C}$ *is transverse Riemann-Lorentz*. Let $(M, \mathcal{C})$ be a transverse Riemann-Lorentz conformal manifold. We say that $g \in \mathcal{C}$ *is conformally* II*-flat if* $II_{\Sigma}^{R} = hg_{\Sigma}$*, for some radical vectorfield* R *and some* $h \in C^{\infty}(\Sigma)$. This definition does not depend on R and, even more, it is conformal: if $\overline{g} = e^{2f}g \in \mathcal{C}$, then it holds

$$\overline{II}_{\Sigma}^{R} = e^{2f}\left\{II_{\Sigma}^{R} - Rf|_{\Sigma}\, g_{\Sigma}\right\} \tag{1}$$

Thus we say that $(M, \mathcal{C})$ is *conformally* II*-flat if some (and thus, any) metric* $g \in \mathcal{C}$ *is conformally* II*-flat*.

Proposition 7 *A transverse Riemann-Lorentz conformal manifold* $(M, \mathcal{C})$ *is conformally* II*-flat if and only if around every singular point* $p \in \Sigma$ *there exist an open neighbourhood* $\mathbb{U}$ *in* M *and a metric* $g \in \mathcal{C}$ *which is* II*-flat on* $\mathbb{U}$*, that is* $II_{\Sigma\cap\mathbb{U}} = 0$.

Proof: Let $(\mathbb{U}, E)$ be an adapted orthonormal frame near $p \in \Sigma$ (that is, E_m is radical and $(E_1, ..., E_{m-1})$ are orthonormal) and $g \in \mathcal{C}$. If $\mathcal{C}$ is conformally II-flat, then there exists $h \in C^{\infty}(\Sigma)$ such that $II_{\Sigma}^{E_m} = hg_{\Sigma}$. Take $\widehat{h} \in C^{\infty}(\mathbb{U})$ any local extension of h (shrinking $\mathbb{U}$ if necessary). There exists $f \in C^{\infty}(\mathbb{U})$ (shrinking again $\mathbb{U}$ if necessary) satisfying $E_m f = \widehat{h}$ (since it is locally a first order linear equation), what gives on $\mathbb{U}$: $II_{\Sigma}^{E_m} = (E_m f)|_{\Sigma}\, g_{\Sigma}$. Let $\widehat{f} \in C^{\infty}(M)$ be any extension of (possibly a restriction of) f. Applying (1) to g and $\overline{g} := e^{2\widehat{f}}g \in \mathcal{C}$ we have $\overline{II}_{\Sigma}^{E_m} = 0$.

To show the converse we start considering $g \in \mathcal{C}$. Since conformally II-flatness is a local condition, it suffices to take an arbitrary $p \in \Sigma$ and $\overline{g} = e^{2\widehat{f}}g \in \mathcal{C}$ such that $\overline{g}$ is II-flat around p. Then, formula (1) applied to g and $\overline{g}$ shows that $II_p^{\xi} = (\xi f)\, g_p$, where $\xi \in Rad_p(M) - \{0\}$ ♣

In what follows, we study *conformally* II*-flat Riemann-Lorentz conformal* structures with transverse radical. Let g and $\overline{g} = e^{2f}g \in \mathcal{C}$ be two transverse Riemann-Lorentz metrics which are II-flat. Formula (1) shows that $(Rf)|_{\Sigma} = 0$. The expression of $grad_g(f)$ in an adapted orthonormal frame such that $R = E_m$ is $grad_g(f) = \sum_{i=1}^{m-1}(E_i f) E_i + \tau^{-1}(Rf) R$, thus $grad_g(f)$ extends to the whole M. Now a simple computation gives

$$\overline{III}^R = e^{2f}\left\{III^R - II^R\left(grad_g(f), R\right) g_{\Sigma}\right\} \tag{2}$$

We say that $g \in \mathcal{C}$ is *conformally III-flat if it is II-flat (in order that III^R exists) and it holds $III^R = kg_{\Sigma}$, for some radical vectorfield R and some $k \in C^{\infty}(\Sigma)$.* Since II-flatness is not conformal, the above definition, although independent of R, cannot be conformal. However, it is conformal in the subset of II-flat metrics.

Definition 8 *We say that a transverse Riemann-Lorentz conformal manifold $(M, \mathcal{C})$ with transverse radical is conformally III-flat if it is conformally II-flat and every $g \in \mathcal{C}$ which is II-flat on some open $\mathbb{U}$ of M is also conformally III-flat on $\mathbb{U}$.*

Note that there may exist no conformally III-flat metrics on a conformally III-flat manifold, simply because there may exist no II-flat metric there. However, since a conformally III-flat space is conformally II-flat, we deduce from Proposition 7 that there always exist locally II-flat metrics. Let us show that in fact there also exist locally III-flat metrics:

Proposition 9 *A transverse Riemann-Lorentz conformal manifold $(M, \mathcal{C})$ with transverse radical is conformally III-flat if and only if around every singular point $p \in \Sigma$ there exist an open neighbourhood $\mathbb{U}$ in M and a metric $g \in \mathcal{C}$ which is III-flat on $\mathbb{U}$, that is $III_{\Sigma \cap \mathbb{U}} = 0$.*

Proof: Consider $p \in \Sigma$ and $(\mathbb{U}, E)$ a *completely adapted orthonormal frame* (i.e., E_m is radical and $(E_1, ..., E_{m-1})$ are orthonormal and tangent to Σ). If $(M, \mathcal{C})$ is conformally III-flat, there exist $g \in \mathcal{C}$ which is II-flat on $\mathbb{U}$ (without loss of generality) and $k \in C^{\infty}(\Sigma \cap \mathbb{U})$, such that $III^{E_m} = kg_{\Sigma}$. Since the radical is transverse, we have $II^{E_m}_{mm} \neq 0$, thus $k_1 := \frac{k}{II^{E_m}_{mm}}$ is C^{∞} on $\Sigma \cap \mathbb{U}$. As in Proposition 7 we can obtain $f \in C^{\infty}(\mathbb{U})$ such that $E_m f = \tau \widehat{k}_1$, where $\tau = g(E_m, E_m)$ and $\widehat{k}_1 \in C^{\infty}(\mathbb{U})$ is any local extension of k_1. Since $(E_m f)|_{\Sigma} = 0$, we get $grad_g(f) \in \mathfrak{X}(\mathbb{U})$ and we have $II^{E_m}(grad_g(f), E_m) = (\tau^{-1} E_m f)_{\Sigma} II^{E_m}_{mm} = k$. Now, take any extension $\widehat{f} \in C^{\infty}(M)$ of (possibly a restriction of) f. Since g is II-flat, we deduce from (1) that $\overline{g} = e^{2\widehat{f}}g \in \mathcal{C}$ is also II-flat on $\mathbb{U}$. We also deduce that $\overline{g}$ is III-flat on $\mathbb{U}$.

To prove the converse, first observe that the hypothesis implies in particular that $(M, \mathcal{C})$ is conformally II-flat. Consider $p \in \Sigma$ and $g \in \mathcal{C}$, II-flat on a neighbourhood of p. By hypothesis, there exists $\overline{g} = e^{2f}g \in \mathcal{C}$ which is III-flat around p. Thus we deduce from (2) that $III^R = II^R\left(grad_g\left(f\right), R\right)g_\Sigma$, so g is conformally III-flat ♣

In what follows we shall assume that $\dim M = m \geq 4$. We now study the extendibility of *the Weyl tensor*, naturally defined on $(\mathbb{M}, \mathcal{C}_\mathbb{M})$. It is well-known that this tensor plays a main role in deciding when $\mathbb{M}$ is (locally) conformally flat, according to *Weyl Theorem: a pseudoriemannian conformal manifold is (locally) conformally flat if and only if the Weyl tensor vanishes identically* (see for instance the preliminaries of [3]). At the end of the paper we discuss the problem of establish a modified version of Weyl Theorem for transverse Riemann-Lorentz conformal manifolds.

The *Weyl tensor* W on $(\mathbb{M}, g_\mathbb{M})$ can be defined as

$$W := K - h \bullet g \in \mathcal{I}_4^0\left(\mathbb{M}\right),$$

where $h = \frac{1}{m-2}\left\{Ric - \frac{Sc}{2(m-1)}g\right\}$ is the *Schouten tensor*, Ric is the Ricci tensor and Sc is the scalar curvature associated to $(\mathbb{M}, g_\mathbb{M})$, and where

$$\bullet : S^2\left(\mathbb{M}\right) \times S^2\left(\mathbb{M}\right) \to \mathcal{I}_4^0\left(\mathbb{M}\right)$$

is the so-called *Kulkarni-Nomizu product*, given by

$$\theta \bullet \omega\left(x, y, z, t\right) := \det\begin{pmatrix} \theta\left(x, z\right) & \omega\left(x, t\right) \\ \theta\left(y, z\right) & \omega\left(y, t\right) \end{pmatrix} + \det\begin{pmatrix} \omega\left(x, z\right) & \theta\left(x, t\right) \\ \omega\left(y, z\right) & \theta\left(y, t\right) \end{pmatrix}$$

If we pick $\overline{g} = e^{2f}g \in \mathcal{C}$, then the Weyl tensor associated to $(\mathbb{M}, \overline{g}_\mathbb{M})$ satisfies $\overline{W} = e^{2f}W$, thus *the Weyl conformal curvature* $\mathcal{W} :=\uparrow_2^1 W \in \mathcal{I}_3^1\left(\mathbb{M}\right)$ becomes a conformal invariant. Notice that the extendibility of W (which is equivalent to the extendibility of $\mathcal{W}$) is a conformal condition, therefore it should be stated in terms of the conformal structure. In fact, we prove that it is equivalent to conformal III-flatness.

Theorem 10 *Let $(M, \mathcal{C})$ be a transverse Riemann-Lorentz conformal manifold, with $\dim M = m \geq 4$. Then W (smoothly) extends to the whole M if and only if the radical is transverse and $\mathcal{C}$ is conformally III-flat.*

Proof: If $(M,\mathcal{C})$ has transverse radical and is conformally III-flat, there exist (Proposition 9) a M-open covering $\{\mathbb{U}_\alpha\}$ of Σ and a family of metrics $\{g_\alpha\}$ in $\mathcal{C}$ such that g_α is III-flat on $\mathbb{U}_\alpha$. By Theorem 1, the covariant curvature K_α, the Ricci tensor Ric_α and the scalar curvature Sc_α associated to g_α extend to $\Sigma\cap\mathbb{U}_\alpha$, therefore the Weyl tensor W_α also extends to $\Sigma\cap\mathbb{U}_\alpha$. Since this is a conformal condition, W_α extends to $\Sigma\cap\mathbb{U}_\beta$ for all β, and thus W_α extends to the whole M.

To show the converse we start picking an adapted orthonormal frame $(\mathbb{U},E)$. Then, we can express the functions $W_{abcd}=W\left(E_a,E_b,E_c,E_d\right)$ as second order polynomials in $\tau^{-1}=\left(g\left(E_m,E_m\right)\right)$. Let us call $\left(W_{abcd}\right)_0,\left(W_{abcd}\right)_1$ and $\left(W_{abcd}\right)_2$ the differentiable coefficients of the terms of order 0, 1 and 2. Since $\tau=0$ is a local equation for Σ, W extends to $\mathbb{U}$ if and only if the restricted functions $\left.\left(W_{abcd}\right)_2\right|_\Sigma$ and $\left(W_{abcd}\right)_1+\tau^{-1}\left.\left(W_{abcd}\right)_2\right|_\Sigma$ identically vanish.

Suposse the radical is tangent to Σ at a singular point $p\in\Sigma$. We can choose the frame such that $E_1(p),E_2(p)\in T_pM-T_p\Sigma$. But then, using that $II^{E_m}\left(E_m,E_m\right)(p)=0$ (because the radical is tangent), we obtain $\left(W_{1323}(p)\right)_2=\frac{\varepsilon_3}{m-2}II_p^{E_m}\left(E_1,E_m\right)II_p^{E_m}\left(E_2,E_m\right)$. Since E_1 and E_2 are transverse to Σ at p, $\left(W_{1323}(p)\right)_2\neq0$, hence W cannot be extended. Therefore the radical must be transverse to Σ.

Once we know that the radical must be always transverse to Σ (thus $II_{mm}^{E_m}\neq0$), we can choose the orthonormal frame $(\mathbb{U},E)$ completely adapted. Thus, picking i,j,k different from m, with i,j different from k, and using that $II_{im}^{E_m}=0$, we have: if $i\neq j$, then $0=\left.\left(W_{ikjk}\right)_2\right|_\Sigma=-\frac{\varepsilon_k}{m-2}II_{ij}^{E_m}II_{mm}^{E_m}$. Since $II_{mm}^{E_m}\neq0$, we get $II_{ij}^{E_m}=0$. If $i=j$ (and using $II_{ij}^{E_m}=0$), the $\begin{pmatrix} m-1\\ 2\end{pmatrix}$ equalities $0=\left.\left(W_{ikik}\right)_2\right|_\Sigma$, suitably manipulated, give us $\varepsilon_iII_{ii}^{E_m}+\varepsilon_kII_{kk}^{E_m}=\frac{2C}{m-1}$, where $C=\sum_{l=1}^{m-1}\varepsilon_lII_{ll}^{E_m}\in C^\infty(\mathbb{U})$. Substracting the equation for i,k from the equation for k,j, we obtain $\varepsilon_iII_{ii}^{E_m}-\varepsilon_jII_{jj}^{E_m}=0$, thus $\varepsilon_iII_{ii}^{E_m}=\varepsilon_jII_{jj}^{E_m}$. Defining $k:=\varepsilon_1II_{11}^{E_m}\in C^\infty(\Sigma\cap\mathbb{U})$, it holds $II_{ii}^{E_m}=\varepsilon_i\varepsilon_1II_{11}^{E_m}=kg_{ii}$ and $II_{ij}^{E_m}=0=kg_{ij}$ (where $i\neq j$), what means $II_\Sigma^{E_m}=kg_\Sigma$, that is, g is conformally II-flat on $\mathbb{U}$, and therefore $(M,\mathcal{C})$ is conformally II-flat.

Once we know that $(M,\mathcal{C})$ is conformally II-flat, we can choose a metric $g\in\mathcal{C}$ which is II-flat on $\mathbb{U}$ (shrinking $\mathbb{U}$ if neccesary). By Theorem 1, the covariant curvature K associated to g extends to $\Sigma\cap\mathbb{U}$ and, since W also does it, necessarily $h\bullet g$ extends to $\Sigma\cap\mathbb{U}$. Picking i,j,k different from m, with i,j different from k, we get $(h\bullet g)_{ikjk}=\varepsilon_kh_{ij}+\delta_{ij}\varepsilon_ih_{kk}=A_{ijk}+\tau^{-1}B_{ijk}$, therefore the function

$$B_{ijk} := \frac{1}{m-2}\left\{\varepsilon_k K_{imjm} + \delta_{ij}\varepsilon_i K_{kmkm} - \frac{2\varepsilon_k\delta_{ij}\varepsilon_i}{m-1}\sum_{l=1}^{m-1}\varepsilon_l K_{lmlm}\right\}$$

must vanish on Σ. Using the same argument as before, but with the equalities $0 = B_{ijk}|_{\Sigma}$, we get $III^{E_m} = kg_{\Sigma}$, where $k := \varepsilon_1 III^{E_m}_{11} \in C^{\infty}(\Sigma \cap \mathbb{U})$, that is g is conformally III-flat on $\mathbb{U}$, and thus $(M, \mathcal{C})$ is conformally III-flat ♣

Let us consider the following conjecture:

Conjecture 11 *Let $(M, \mathcal{C})$ be a transverse Riemann-Lorentz conformal manifold, with $\dim M = m \geq 4$. A necessary condition for being $W = 0$ is that, around every singular point $p \in \Sigma$, there exist a coordinate system $(\mathbb{U}, x)$ and a metric $g \in \mathcal{C}$ such that $g = \sum_{i=0}^{m-1}(dx^i)^2 + \tau(dx^m)^2$, where $\tau = 0$ is a local equation for Σ.*

Using Theorem 6, it becomes obvious that the necessary condition stated in the conjecture is always sufficient for having $W = 0$ around Σ.

If the conjecture is true, Σ must be (locally) conformally flat, which is well known equivalent to either $W^{\Sigma} = 0$ (if $m > 4$) or $\nabla^{\Sigma}_X h^{\Sigma}(Y, Z) = \nabla^{\Sigma}_Y h^{\Sigma}(X, Z)$ (if $m = 4$). But the extendibility of W, equivalent (Theorem 10) to conformal III-flatness, implies (Proposition 9) the existence of a metric $g \in \mathcal{C}$ which is III-flat around Σ, thus satisfying (Proposition 4):

$$W\mid_{T\Sigma} = (K - h\bullet g)\mid_{T\Sigma} = K^{\Sigma} - h\mid_{T\Sigma}\bullet g_{\Sigma} = W^{\Sigma} + (h^{\Sigma} - h\mid_{T\Sigma})\bullet g_{\Sigma} \quad .$$

Because conditions $W = 0$ and $W^{\Sigma} = 0$ are conformal, any counterexample $(M, \mathcal{C})$ to the above conjecture must admit a metric $g \in \mathcal{C}$ which is III-flat around Σ and satisfies either $h^{\Sigma} \neq h\mid_{T\Sigma}$ (if $m > 4$) or (Lemma 3) $\nabla_X h(Y, Z) \neq \nabla_Y h(X, Z)$, for some $X, Y, Z \in \mathfrak{X}(\Sigma)$ (if $m = 4$). Now a straightforward computation for III-flat metrics, using an orthonormal completely adapted frame, leads to the following expression in terms of extendible quantities:

$$\begin{aligned} h^{\Sigma}_{ij} - h_{ij}\mid_{T\Sigma} &= \frac{-1}{m-2}\left\{\frac{K_{imjm}}{\tau} - \frac{1}{m-3}\sum_{l=1}^{m-1}K_{iljl} - \right. \\ &\left. -\frac{1}{m-1}\left[\sum_{k=1}^{m-1}\frac{K_{kmkm}}{\tau} - \frac{1}{m-3}\sum_{k,l=1}^{m-1}K_{klkl}\right]\delta_{ij}\right\}\mid_{\Sigma} \quad , \end{aligned}$$

$(i,j=1,...,m-1)$, which shows that the construction of counterexamples is not easy.

In fact, *the conjecture is true for transverse Riemann-Lorentz warped products*, as we show right now. Let us consider a m-dimensional $(m\geq 4)$ transverse Riemann-Lorentz manifold (M,g) of the form $M=I\times S$, where $\dim I=1$, $0\in I$, and $g=f(t)^2 g_S - tdt^2$, where $f\in C^\infty(I)$, $f>0$ and g_S is riemannian (we identify t, f and g_S with the corresponding pullbacks by the canonical projections). Thus $\Sigma=\{0\}\times S$ is homothetic to S with scale factor $f(0)$. Calling $U\in\mathfrak{X}(M)$ the (nowhere zero) lift of the vectorfield $\frac{d}{dt}\in\mathfrak{X}(I)$, one inmediately sees that U is radical and transverse to Σ. It is not difficult to compute the curvature tensors on $\mathbb{M}$. Standar results on warped products (see [8], Chapter 7) lead to (we denote by $X,Y\in\mathfrak{X}(M)$ the lifts of corresponding vectorfields $\overline{X},\overline{Y}\in\mathfrak{X}(S)$) $\nabla_U U=\frac{1}{2t}U$, $\nabla_U X=\nabla_X U=\frac{f'}{f}X$ and $\nabla_X Y=g(X,Y)\frac{f'}{tf}U+\nabla^S_{\overline{X}}\overline{Y}$ (where ∇^S is the Levi-Civita connection on S and $\nabla^S_{\overline{X}}\overline{Y}$ is the lift of the corresponding vectorfield on S) and also the following expressions for the curvature tensors:

$$\begin{cases} K=f^2K^S+\frac{f'^2f^2}{2t}g_S\bullet g_S+\frac{f}{2}(\frac{f'}{t}-2f'')g_S\bullet dt^2 \\ Ric=Ric^S-\left(\frac{f}{2t}(\frac{f'}{t}-2f'')-(m-2)\frac{f'^2}{t}\right)g_S+\frac{m-1}{2f}(\frac{f'}{t}-2f'')dt^2 \\ Sc=\frac{Sc^S}{f^2}-\frac{m-1}{f^2}\left(\frac{f}{t}(\frac{f'}{t}-2f'')-(m-2)\frac{f'^2}{t}\right) \\ h=\frac{m-3}{m-2}h^S+\left(\frac{Sc^S}{2(m-2)^2(m-1)}+\frac{f'^2}{2t}\right)g_S+ \\ \qquad\qquad +\left(\frac{tSc^S}{2(m-1)(m-2)f^2}+\frac{1}{2f}(\frac{f'^2}{f}+\frac{f'}{t}-2f'')\right)dt^2 \\ W=f^2W^S+\frac{1}{(m-2)}\left(Ric^S-\frac{Sc^S}{m-1}g_S\right)\bullet\left(\frac{f^2}{m-3}g_S+tdt^2\right) \end{cases}$$

(K^S, Ric^S, Sc^S, h^S and W^S denote of course the pullbacks by the projection of the corresponding tensor fields on S). It follows:

Lemma 12 *The following three conditions are equivalent: (1) K extends to M, (2) $f'(0)=0$ and (3) h extends to M. Also the following are equivalent: (1) Ric extends to M, (2) $(f'/t)(0)=0$ and (3) Sc extends to M. Moreover, W extends to M in any case.*

The fact that W extends to M was obvious from the very beginning: the map $\Psi\equiv\psi\times id:(I-\{0\})\times S\to\mathbb{R}\times S$, given by $T\equiv\psi(t):=\int_0^t\frac{|s|^{\frac{1}{2}}ds}{f(s)}$, is a conformal diffeomorphism onto its (non-connected) image with the metric $\overline{g} \equiv -(dT)^2 + g_S$, thus it preserves the $\binom{1}{3}$-Weyl tensors, and since $\overline{g}$ is regular around $T = 0$ and $f(0) \neq 0$, $\overline{W}$ (and therefore W) extends to the whole M. It follows from Theorem 10 that the conformal manifold $(M, [g])$ is (in any case) conformally III-flat.

Lemma 13 *The following four conditions are equivalent: (1) $W = 0$, (2) $W^S = 0 = Ric^S - \frac{Sc^S}{m-1} g_S$ and (3) Σ has constant (sectional) curvature.*

Proof: (1) $\Leftrightarrow$ (2) follows from the above formula. (2) $\Rightarrow$ (3): $Ric^S - \frac{Sc^S}{m-1} g_S = 0$ implies (Schur's lemma) $Sc^S = (m-1)(m-2)C$ (constant), thus $h^S = \frac{C}{2} g_S$; moreover $W^S = 0$ leads to $K^S = \frac{C}{2} g_S \bullet g_S$. (3) $\Rightarrow$ (2): From $K^S = \frac{C}{2} g_S \bullet g_S$, one immediately gets $W^S = 0 = Ric^S - \frac{Sc^S}{m-1} g_S$ ♣

Proposition 14 *The Conjecture 11 is true for any transverse Riemann-Lorentz conformal manifold $(M, \mathcal{C})$ such that some $g \in \mathcal{C}$ is a warped product.*

Proof: Let $g = f(t)^2 g_S - t dt^2 \in \mathcal{C}$ be a transverse warped product metric on $M = I \times S$. Note that $g = f(t)^2 \left\{ g_S - \frac{t}{f(t)^2} dt^2 \right\}$. From $W = 0$ and Lemma 13 we get, around any $p \in \Sigma$, coordinates $(\mathbb{V}, y)$ of Σ such that $f(0)^2 g_S = g_\Sigma = e^{2h} \sum_{i=1}^{m-1} (dy^i)^2$, for some $h \in C^\infty(\Sigma)$. Choosing $x^i := y^i \circ \pi$, $x^m := t$ and $\tau := \frac{-te^{-2h}}{f(t)^2}$, we get $g = e^{2h} f(t)^2 \left\{ \sum_{i=1}^{m-1} (dx^i)^2 + \tau (dx^m)^2 \right\}$, and we are finished ♣

References

[1] E. Aguirre and J. Lafuente. *Trasverse Riemann-Lorentz metrics with tangent radical*. Diff. Geom. its App, 24, 2, 91-100, 2005.

[2] J. B. Hartle and S. W. Hawking. *Wave Function of the Universe*. Phys. Rev., D41, 1815-34, 1990.

[3] U. Hertrich-Jeromin. *Introduction to Möbius Differential Geometry*. Cambridge Univ. Press, 2003.

[4] M. Kossowski. *Fold singularities in pseudoriemannian geodesic tubes*. Proc. Amer. Math. Soc., 95, 463-469,1985.

[5] M. Kossowski. *Pseudo-riemannian metric singularities and the extendability of parallel transport*. Proc. Amer. Math. Soc., 99, 147-154, 1987.

[6] M. Kossowski and M. Kriele. *Transverse, type changing, pseudo riemannian metrics and the extendability of geodesics*. Proc. R. Soc. Lond. A 444, 297-306,1994.

[7] M. Kossowski and M. Kriele. *The volume blow-up and characteristic classes for transverse, type changing, pseudo-riemannian metrics.* Geom. Dedicata 64, 1-16. 1997.

[8] B. O'Neill. *Semi-riemannian Geometry.* Academic Press, 1983.